which is again a hyperbola and has as asymptotes the lines

$$\frac{Y}{X} = \frac{-(r_1 - r_2 \cos\theta_2)}{r_2 \sin\theta_2} = \frac{-h}{R} \tag{16}$$

and

$$\frac{Y}{X} = \frac{(r_1 + r_2 \cos\theta_2)}{r_2 \sin\theta_2} = \frac{2r_1 - h}{R} \tag{17}$$

The first asymptote is the line $\langle P_1P_2\rangle$, as found before. However, the vertical axis is not an asymptote in this case. Consequently, the direction for which minimum speed is obtained is different from the common result of the Newtonian and uniform fields.

Coordinate Perturbations from Kepler Orbits

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1. Introduction

IN the astrodynamics literature various perturbation analyses in terms of geometrically obvious coordinates have enjoyed some popularity in connection with guidance work. In particular, one is concerned with the perturbations resulting from position and velocity errors at certain points along the orbit, which sometimes are referred to as "orbit sensitivities." These formulations contrast somewhat with the classical perturbation analyses in terms of the osculating elements or other sets of canonical variables.

At an earlier occasion the author examined the various perturbations of a satellite orbit in terms of a moving coordinate system (Fig. 1) centered at the nominal instantaneous satellite position 0′. The position and velocity of the vehicle in terms of ξ, η, ζ describe its actual motion relative to the "nominal" orbit.† The latter may have been the aim of an imperfect guidance maneuver or actually may have existed prior to whatever physical disturbances brought the discrepancies ξ, η, ζ into being.

A study of satellite orbits according to this approach yielded first-order perturbations of near-circular paths.[1, 2] Ever since that time an interest has persisted in the extension of this formulation to orbits with large eccentricities. The following note records the solutions of the homogenous form of the governing differential equations for any elliptic, near-parabolic, or hyperbolic orbit.

In keeping with the spirit of the earlier work, these solutions were derived from the simple geometric facts that they represent. To be sure, one could extract the same results from the more sophisticated and geometrically more remote formalism of Hamiltonian mechanics.‡ The use of canonical transformations becomes unavoidable in progressing from the present complementary solutions to particular ones if the algebra is to remain tolerable.

The complementary solutions themselves, as given in this note, may be used to study the divergence of transfer orbits from the earth to the moon or to another planet due to guidance errors. They will also describe the spread in descent trajectories to the lunar surface or re-entry in the outer fringes of the earth's atmosphere due to anomalies in vehicle position or velocity at the start of such maneuvers. In many applications an accurate description of perturbative displacements cannot be given without including the sustained effects of drag, a nonspherical central body, and extraneous gravitation. The necessary extensions of the present formulation are currently under study; they can be regarded as a logical refinement of the popular patched-conic approach to multiphase orbits.

2. Elliptic Orbits of Arbitrary Eccentricity

The equations of motion in terms of ξ, η, ζ have been derived for nonvanishing eccentricity in Ref. 1 and will be merely restated here:

$$\xi'' - 2\eta' - \xi - 2[\xi + e(\xi' - \eta)\sin f]/(1 + e\cos f) = 0 \tag{1}$$

$$\eta'' + 2\xi' - \eta - [-\eta + 2e(\eta' + \xi)\sin f]/(1 + e\cos f) = 0 \tag{2}$$

$$\zeta'' + [\zeta - 2\zeta' e\sin f]/(1 + e\cos f) = 0 \tag{3}$$

where the primes denote derivatives with respect to the true anomaly f of the nominal orbit.

At first sight these equations appear sufficiently awkward to be discouraging; but one notes that they apply equally well to elliptic and hyperbolic orbits, and, with $e = 1$, they are valid for parabolic orbits. In the derivation of these equations the usual linearizations were employed, and hence their solutions must be regarded as a first-order representation of the perturbations in a geometric sense. It stands to reason that such solutions should be obtainable from the first derivatives of the position in a Kepler orbit with respect to the orbit parameters (linear sensitivities). This is the rationale to be followed below.

Remembering the basic formulas of an elliptic orbit, one has

$$r = \frac{a(1-e^2)}{1 + e\cos f} \qquad x = r\cos f \qquad y = r\sin f$$

or, in terms of the eccentric anomaly,

$$r = a(1 - e\cos E) \qquad x = a(\cos E - e)$$
$$y = a(1 - e^2)^{1/2}\sin E$$

The x axis points from the focus through pericenter, and the y axis coincides with $f = \pi/2$. From Kepler's equation, one obtains

$$\begin{aligned}\frac{\partial E}{\partial a} &= -\frac{3}{2a}\frac{E - e\sin E}{1 - e\cos E}\\ \frac{\partial E}{\partial e} &= \frac{\sin E}{1 - e\cos E}\\ \frac{\partial E}{\partial \tau} &= \frac{(k/a^3)^{1/2}}{1 - e\cos E}\end{aligned} \tag{4}$$

where k is the gravitational constant multiplied by the central mass. Thus $\partial x/\partial a, \ldots, \partial y/\partial\tau$ follow, and if one denotes

$$\delta x = \frac{\partial x}{\partial a}\delta a + \frac{\partial x}{\partial e}\delta e + \frac{\partial x}{\partial \tau}\delta\tau$$

$$\delta y = \frac{\partial y}{\partial a}\delta a + \frac{\partial y}{\partial e}\delta e + \frac{\partial y}{\partial \tau}\delta\tau$$

then

$$\begin{aligned}\xi &= \delta x\cos f + \delta y\sin f\\ \eta &= -\delta x\sin f + \delta y\cos f + r\delta\omega\end{aligned} \tag{5}$$

The δ's represent small changes, and the last term in the expression for η represents a rigid-body rotation of the entire

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† For a later treatment of a similar approach, see Ref. 3.

‡ In fact, some development in this direction from Brouwer's perturbation method has been reported to the author by D. Brouwer and H. R. Westerman.

ellipse in the orbit plane due to the error $\delta\omega$ in the argument of pericenter. If one writes (5) out in detail, using f as the independent variable, and represents

$$\int_0^f \frac{a^{3/2}\,(1-e)^{3/2}}{k^{1/2}(1+e\cos f)^2}\,df$$

as $(t-\tau)$, one obtains

$$\xi = \delta a\left[\frac{1-e^2}{1+e\cos f} - \frac{3e}{2}\left(\frac{k}{(1-e^2)a^3}\right)^{1/2}(t-\tau)\sin f\right] - \delta e\, a\cos f - \delta\tau\, e\left(\frac{k}{(1-e^2)a}\right)^{1/2}\sin f \quad (6)$$

$$\eta = -\delta a\,\frac{3}{2}\left(\frac{k}{(1-e^2)a^3}\right)^{1/2}(1+e\cos f)(t-\tau) + \delta e\, a\sin f\,\frac{(2+e\cos f)}{1+e\cos f} - \delta\tau\left(\frac{k}{(1-e^2)a}\right)^{1/2}(1+e\cos f) + \delta\omega\,\frac{a(1-e^2)}{1+e\cos f} \quad (7)$$

The constants δa, δe, $\delta\tau$, $\delta\omega$ in these expressions retain their geometric meaning but will, in practice, be determined from the (initial) conditions of the perturbed motion at some point f_0, t_0. Thus,

$$\delta a\left[\frac{1-e^2}{1+ecf} - \frac{3e}{2}(1-e^2)^{-1/2}\,sf\,\bar{t}\right] + \delta e\,[-acf] + \delta\tau\left[-e\left(\frac{k}{a(1-e^2)}\right)^{1/2} sf\right] = \xi_0$$

$$\delta a\left[-\frac{3}{2}(1-e^2)^{-1/2}(1+ecf)\bar{t}\right] + \delta e\left[\frac{asf\,(2+ecf)}{1+ecf}\right] + \delta\tau\left[-\left(\frac{k}{a(1-e^2)}\right)^{1/2}(1+ecf)\right] + \delta\omega\left[\frac{a(1-e^2)}{1+ecf}\right] = \eta_0$$

$$\delta a\left[-\frac{e(1-e^2)sf}{2(1+ecf)^2} - \frac{3e\,cf\,\bar{t}}{2(1-e^2)^{1/2}}\right] + \delta e\,[asf] + \delta\tau\left[-e\left(\frac{k}{a(1-e^2)}\right)^{1/2} cf\right] = \xi_0' \quad (8)$$

$$\delta a\left[-\frac{3(1-e^2)}{2(1+ecf)} + \frac{3e\,sf\,\bar{t}}{2\,(1-e^2)^{1/2}}\right] + \delta e\left[\frac{2a\,cf}{1+ecf} + \frac{ae(1+ec^3f)}{(1+ecf)^2}\right] + d\tau\left[e\left(\frac{k}{a(1-e^2)}\right)^{1/2} sf\right] + \delta\omega\left[\frac{ae(1-e^2)\,sf}{(1+ecf)^2}\right] = \eta_0'$$

where cf and sf are abbreviations for $\cos f$ and $\sin f$, $\bar{t} = (k/a^3)^{1/2}(t-\tau)$, and the symbol f is understood to represent f_0.

The inversion of this system of algebraic equations in the general case presents excessive labor. In practice, one would do this numerically for any given set of values for a, e, τ, f_0, t_0, ξ_0, η_0, ξ_0', and η_0'. An explicit inversion is conveniently possible for the case of $f_0 = 0$, i.e., with the initial point at the nominal pericenter. In that case (6) and (7) become

$$\xi = \frac{2}{(1-e)^2}\,[\eta_0'(1+e) + \xi_0(2+e)]\left\{\frac{1-e^2}{1+e\cos f} - \frac{3e}{2}\sin f\left(\frac{k}{(1-e^2)a^3}\right)^{1/2}(t-t_0)\right\} - \frac{1+e}{1-e}\,[2\eta_0' + 3\xi_0]\cos f + \xi_0'\sin f \quad (9)$$

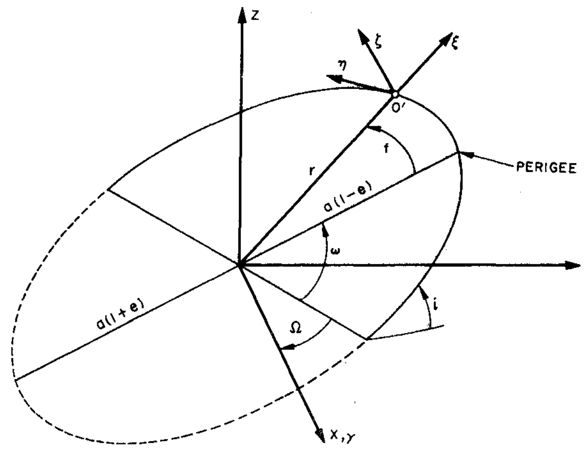

Fig. 1 Definitions of moving coordinates, ξ, η, ζ.

$$\eta = \frac{-3}{(1-e)^2}\,[\eta_0'(1+e) + \xi_0(2+e)] \times (1+e\cos f)\left(\frac{k}{(1-e^2)a^3}\right)^{1/2}(t-t_0) + \frac{1+e}{1-e}\,[2\eta_0' + 3\xi_0] \times \sin f\,\frac{(2+e\cos f)}{1+e\cos f} + \eta_0\,\frac{(1+e)}{1+e\cos f} + \frac{\xi_0'}{e} \times \left[1 + e\cos f - \frac{(1+e)^2}{1+e\cos f}\right] \quad (10)$$

As $e \to 0$, these expressions reduce to the near-circular case treated in Ref. 1.

Finally, the solution $\zeta(f)$ of (3) remains to be found. Here, too, one could proceed with a geometric derivation based on small changes δi and $\delta\Omega$ in the inclination and nodal angle of the orbit plane. However, it is convenient to use a more direct approach in terms of ζ_0 and ζ_0'.

If r_0 is the initial radius at f_0, then a displacement ζ_0 normal to the orbit plane implies a rotation of the actual plane of motion by an angle ζ_0/r_0 about the line $f_0 - (\pi/2)$. If, however, the total velocity vector v_0 at f_0 is to remain parallel to itself, as the condition $\zeta \neq \phi$, $(d\zeta/dt)_{f_0} \equiv \dot{\zeta}_0 = 0$ implies, then it can be shown that a simultaneous rotation $-[e\sin f_0\,\zeta_0/(1 + e\cos f_0)]$ about r_0 must be applied. Finally, an initial value of $\dot{\zeta}_0$, different from zero, would represent a rotation $\dot{\zeta}_0/\dot{f}_0 r_0 = \zeta_0'/r_0$ about r_0. Developing $\zeta(f)$ explicitly as a superposition of these small-angle rotations, one finds

$$\zeta = \frac{1}{1+e\cos f}\{\zeta_0\,[\cos(f-f_0) + e\cos f] + \zeta_0'\sin(f-f_0)(1+e\cos f_0)\} \quad (11)$$

which represents the solution of (3). This expression is valid for elliptic, hyperbolic, and parabolic orbits, since the form of the nominal orbit did not enter into its derivation. In terms of the orbit elements, this result can be written as

$$\zeta = \frac{2p}{1+e\cos f}\,[\delta i\sin\theta - \delta\Omega\sin i\cos\theta] \quad (11a)$$

where $2p = a(1-e^2)$ or $a(e^2-1)$ for elliptic or hyperbolic orbits, respectively.

3. Hyberbolic Orbits

For hyperbolic orbits, one may proceed in perfect analogy with the previous section to obtain

$$\xi = \delta a\left[\frac{e^2-1}{1+e\cos f} - \frac{3e}{2}\left(\frac{k}{(e^2-1)a^3}\right)^{1/2}(t-\tau)\sin f\right] + \delta e\, a\cos f - \delta\tau\, e\left(\frac{k}{(e^2-1)a}\right)^{1/2}\sin f \quad (12)$$

$$\eta = -\delta a \frac{3}{2}\left(\frac{k}{(e^2-1)a^3}\right)^{1/2}(1+e\cos f)(t-\tau) - \delta e\, a \sin f \frac{(2+e\cos f)}{1+e\cos f} - \delta\tau\left(\frac{k}{(e^2-1)a}\right)^{1/2}\times (1+e\cos f) + \delta\omega\frac{a(e^2-1)}{1+e\cos f} \quad (13)$$

4. Parabolic Orbits

In treating the perturbations about a parabolic orbit, one notes that one cannot simply let $e \to 1$ in (6) and (7), due to the divisors $1 - e^2$; and, moreover, $a \to \infty$. However, the quantity $a(1 - e^2)$ remains bounded and frequently is denoted by $2p$. In order to rearrange the previous results to read in terms of p, one notes that $\delta a(1 - e^2) = 2\delta p + 2ae\ \delta e$. However, for the parabolic case one may write $e = 1 + \epsilon$, $\delta e = \epsilon$, and $1 - e^2 = -2\epsilon + 0(\epsilon^2)$, so that $\delta a(1 - e^2) = 2\delta p - 2p\ (1 + \epsilon)$. Reformulating (6) and (7) in this fashion and retaining terms of $0(\delta p)$ and $0(\epsilon)$ in place of $0(\delta a)$ and $0(\delta e)$ for the elliptic orbit, one arrives at

$$\xi = \delta p\left\{\frac{2 - \frac{3}{2}\sin^2 f}{1+\cos f} - \frac{1}{2}\sin^2 f\frac{(1-\cos f)}{(1+\cos f)^2}\right\} + \epsilon p\left\{-\frac{3}{2}(1-\cos f) + \frac{2(1+2\cos f)}{(1+\cos f)^2} + \frac{3}{10}\frac{(1-\cos f)^3}{(1+\cos f)^2}\right\} - \delta\tau\left(\frac{k}{2p}\right)^{1/2}\sin f \quad (14)$$

$$\eta = -\delta p\frac{3}{2}\sin f\left\{1 + \frac{1}{3}\frac{1-\cos f}{1+\cos f}\right\} + \epsilon p\frac{3}{2}\sin f\left\{-\frac{2+\cos f}{1+\cos f} + \frac{\cos f - \frac{1}{3}}{(1+\cos f)^2} + \frac{1}{5}\left(\frac{1-\cos f}{1+\cos f}\right)^2\right\} - \delta\tau\left(\frac{k}{2p}\right)^{1/2}(1+\cos f) + \frac{\delta\omega\, 2p}{1+\cos f} \quad (15)$$

It is evident that one could have obtained the same results by applying the appropriate manipulations to (12) and (13), i.e., by working from the hyperbolic case.

From the definition of ϵ, it is also clear that $\epsilon > 0$ implies a perturbation toward hyperbolic orbits and for $\epsilon < 0$ toward elliptic ones.

5. Applications

As stated in the introduction, the results (6, 7, and 11–15), together with the special cases for $f_0 = 0$ (pericenter) or $f_0 = \pi$ (apocenter), are applicable whenever it is desired to exhibit the departures from an intended orbit due to faulty guidance or orbit determination at some point f_0. Taking the manned lunar mission as an example, one can trace the use of such formulas through its various phases.

Beginning with the coast trajectory that precedes the injection to the parking orbit, one may use (6, 7, and 11) to predict the departures from the nominal position and velocity at the injection point due to perturbations at the termination of launch burn. Next, the special case of these formulas for $e = 0$ [formulas (17–19) of Ref. 1] is useful in exhibiting the response of the parking orbit to errors at injection and, specifically, the discrepancies in position and velocity just prior to launch into the "translunar" phase.

Subsequent to this launch and any of the scheduled midcourse maneuvers, one can use (6, 7, and 11) or (14, 15, and 11) to predict the effects of a guidance error on the lunar approach. In fact, formulas (14) and (15) generally will be preferable to (6) and (7) for near-parabolic orbits (frequently encountered in lunar transfer trajectories) where (6) and (7) begin to lose accuracy. During the lunar approach, a high-precision position determination from the earth can be reduced to position and velocity errors at the last guidance maneuver by means of (14, 15, and 11). This would serve as a summary check on the guidance through midcourse.

After the transition to a lunar parking orbit, the case $e \simeq 0$ prevails again. If the ballistic descent to the lunar surface follows a Hohmann arc one has $f_0 = \pi$, and for the ascent to a return rendezvous in the lunar parking orbit the case $f_0 = 0$ very nearly applies. For the final approach in this rendezvous, the relative motion between the two vehicles can be described in detail by means of (17–19) of Ref. 1. These expressions are simple enough for on-board calculations of corrective maneuvers.

For the earth-bound trajectory from the moon, the earlier remarks on the translunar phase are again applicable. Finally, the expressions (6, 7, and 11) can be used to represent the spread between the high-altitude portions of re-entry trajectories in response to guidance forces exerted at the beginning of this phase. This is especially true for the skip trajectory that may have to be flown in order to reach an alternative landing site. In all of these approach maneuvers the forementioned formulas are helpful during the early part of re-entry, before the aerodynamic forces become comparable to the gravitational attraction.

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Approximate Method for Hypersonic Nonequilibrium Blunt Body Airflows

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The problems of hypersonic airflow including chemical effects have received much attention recently.[1] The inclusion of coupled rate chemistry in the analysis of shock layer flow behind detached shocks has been studied by Hall et al.,[2] who have developed an exact method of solution following the approach of Lick.[3] Starting with a specified bow shock, a numerical integration technique is employed to compute conditions at lattice points within the shock layer, specific streamline shapes being determined from mass flow integrals. The procedure is necessarily involved and demands solution by means of a large high-speed digital computer.

Several approximate methods have been reported of which the streamtube method[4,5] and the shock-mapping method[6] may be mentioned. In the streamtube method the procedure is iterative, an assumed bow shock shape and location and an assumed pressure distribution being improved in successive iterations in real gas flow using continuity equations until a self-consistent solution is obtained. In shock-mapping,

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